STUDY OF BROKEN FUNCTIONS BY COMBINATION OF PSEUDO-LIMITS

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ABSTRACT

We will study the so-called "broken" functions that do not have a correct solution in 0 and whose limits tend towards two distinct values when their variables tend to 0^+ and 0^-. We will use for this a method of combination of pseudo-limits allowing to associate the tendencies towards 0^+ and 0^- variables so as to obtain the exact solution of said broken functions in zero said "neutral", that we will simply note 0. From this way, we will study the limits when x tends to 0 (and not only 0^+ and 0^-) of functions such that x^x, $(e^{\frac{-1}{x^2}})^x$, $(e^{\frac{-1}{x^2}})^{-x}$, $(e^{\frac{-1}{x^2}})^{ax}$, whose results for x tending to 0^+ are respectively 1, 0, $+\infty$, and e^{-a}. We will determine that the solution in 0 (neutral) of these functions is 1 and will extend this case to any function $u(x)^{(v(x))}$ where $x = u(x) = v(x) = 0$. We will show that 0^0 is always equal to 1.

Keywords Zero to the power of zero · indeterminate form · pseudo-limits · broken functions · Non-standard analysis

1 Introduction

Although there is debate about the question [1, 2], the form 0^0 is generally considered indeterminate or context dependent [3, 4].

First, we define the notion of pseudo-limit pair as two limits α and β, for x tending respectively to 0^+ and 0^-, corresponding to a function $f(x)$ having a solution in $x = 0$, of so that:

$$\begin{cases} \alpha = \lim\limits_{h \to 0^+} 0 + h \\ \beta = \lim\limits_{h \to 0^-} 0 + h \\ \gamma = \alpha + \beta = 0 \end{cases}$$

In accordance with these unique prerequisites, α will then be considered as the smallest positive number other than 0 included in $\mathbb{R}$ and β will be considered the largest negative number other than 0 included in $\mathbb{R}$, which further verifies the inequality $\gamma \neq \alpha \neq \beta$.

So, since

$$\alpha + \beta = 0$$

Then

$$\alpha = -\beta$$

And

$$\frac{\alpha}{\beta} = -1$$

In order to continue, we will determine the value:

$$\begin{aligned} \frac{\gamma}{\gamma} \\ \frac{\gamma}{\gamma} &= \frac{\alpha + \beta}{\alpha + \beta} \\ \frac{\alpha}{\alpha + \beta} &= \frac{\gamma}{\gamma} - \frac{\beta}{\alpha + \beta} \\ \frac{\alpha}{\alpha + \beta} &= \frac{\alpha + \beta}{\alpha + \beta} - \frac{\beta}{\alpha + \beta} \\ \frac{\alpha}{\alpha + \beta} &= \frac{\alpha}{\alpha + \beta} \end{aligned}$$

$$\frac{\frac{\alpha}{\alpha+\beta}}{\frac{\alpha}{\alpha+\beta}} = 1 \qquad \frac{\alpha}{\alpha+\beta} = \frac{-\beta}{\alpha+\beta}$$

$$\frac{\frac{1}{\alpha+\beta}}{\frac{1}{\alpha+\beta}} = 1 \qquad \frac{\alpha}{-\beta} = \frac{\alpha+\beta}{\alpha+\beta}$$

$$\frac{\alpha+\beta}{\alpha+\beta} = 1 \qquad -(-1) = \frac{\alpha+\beta}{\alpha+\beta}$$

$$\frac{\gamma}{\gamma} = 1 \qquad 1 = \frac{\alpha+\beta}{\alpha+\beta}$$

$$\frac{\gamma}{\gamma} = 1 \qquad 1 = \frac{\gamma}{\gamma}$$

Insofar as

$$\lim_{t\to 0^+} \frac{t}{t} = 1$$

And

$$\lim_{t\to 0^-} \frac{t}{t} = 1$$

This is not surprising, although it may seem rather unusual. This well-known undefined form $\frac{\gamma}{\gamma}$ has a solution of 1. This is a result that will be found in particular in the study of function x^x. Recall that γ is calculated according to the pseudo-limits α and β.

2 Application to the function x^x

Recall that the limit of x^x for x tending to 0^+ results in 1, our pseudo-limit combination method will have to give the same result for x tending to 0 [5].

We consider :

$$\lim_{x\to\gamma} x^x$$

To determine a solution for x tending to γ, and therefore 0 (neutral), we will start with:

$$\gamma^\gamma = (\alpha+\beta)^{\alpha+\beta}$$
$$\gamma^\gamma = (\alpha+\beta)^{\alpha}(\alpha+\beta)^{\beta}$$
$$\gamma^\gamma = (\alpha+\beta)^{\alpha}(\alpha+\beta)^{-\alpha}$$
$$\gamma^\gamma = \frac{(\alpha+\beta)^{\alpha}}{(\alpha+\beta)\alpha}$$
$$\gamma^\gamma = (\frac{(\alpha+\beta)}{\alpha+\beta})^{\alpha}$$

However

$$\frac{\alpha+\beta}{\alpha+\beta} = 1$$

And

$$1^{\alpha} = 1$$

So

$$\lim_{x \to \gamma} x^x = 1$$

The broken function x^x therefore results in 1 when x tends to 0.

We are now testing our method of combining pseudo-limits on functions $(e^{\frac{-1}{x^2}})^x$, $(e^{\frac{-1}{x^2}})^{-x}$, $(e^{\frac{-1}{x^2}})^{ax}$, whose results for x tending to 0^+ are respectively 0, $+\infty$, and e^{-a}, to see if it is possible to find a result different from that obtained for x tending to 0^+ [6, 7]. If the method works, each limit will result in 1 for x tending to 0 and we will can generalize.

3 Application to the function $(e^{(\frac{-1}{x^2})})^x$

We consider:

$$\lim_{x \to \gamma} (e^{\frac{-1}{x^2}})^x$$

To determine a solution for x tending to γ, and therefore 0 (neutral), we will start with:

$$\begin{aligned}
(e^{\frac{-1}{\gamma^2}})^{\gamma} &= (e^{\frac{-1}{(\alpha+\beta)^2}})^{\alpha+\beta} \\
(e^{\frac{-1}{\gamma^2}})^{\gamma} &= (e^{\frac{-1}{(\alpha+\beta)^2}})^{\alpha}(e^{\frac{-1}{(\alpha+\beta)^2}})^{\beta} \\
(e^{\frac{-1}{\gamma^2}})^{\gamma} &= (e^{\frac{-1}{(\alpha+\beta)^2}})^{\alpha}(e^{\frac{-1}{(\alpha+\beta)^2}})^{-\alpha} \\
(e^{\frac{-1}{\gamma^2}})^{\gamma} &= \frac{(e^{\frac{-1}{(\alpha+\beta)^2}})^{\alpha}}{(e^{\frac{-1}{(\alpha+\beta)^2}})^{\alpha}} \\
(e^{\frac{-1}{\gamma^2}})^{\gamma} &= (\frac{(e^{\frac{-1}{(\alpha+\beta)^2}})}{(e^{\frac{-1}{(\alpha+\beta)^2}})})^{\alpha} \\
(e^{\frac{-1}{\gamma^2}})^{\gamma} &= (\frac{(e^{\frac{1}{(\alpha+\beta)^2}})}{(e^{\frac{1}{(\alpha+\beta)^2}})})^{\alpha} \\
(e^{\frac{-1}{\gamma^2}})^{\gamma} &= ((\frac{e^1}{e^1})^{\frac{1}{(\alpha+\beta)^2}})^{\alpha} \\
(e^{\frac{-1}{\gamma^2}})^{\gamma} &= ((\frac{1}{1})^{(\alpha+\beta)^{-2}})^{\alpha} \\
(e^{\frac{-1}{\gamma^2}})^{\gamma} &= (1^{(\alpha+\beta)^{-2}})^{\alpha} \\
(e^{\frac{-1}{\gamma^2}})^{\gamma} &= 1^{\alpha} \\
(e^{\frac{-1}{\gamma^2}})^{\gamma} &= 1
\end{aligned}$$

The broken function $(e^{\frac{-1}{x^2}})^x$ therefore results in 1 when x tends to 0.

4 Application to the function $(e^{\frac{-1}{x^2}})^{-x}$

We consider:

$$\lim_{x \to \gamma} (e^{\frac{-1}{x^2}})^{-x}$$

To determine a solution for x tending to γ, and therefore 0 (neutral), we will start with:

$$(e^{\frac{-1}{\gamma^2}})^{-\gamma} = \frac{1}{(e^{\frac{-1}{\gamma^2}})^{\gamma}}$$

However

$$(e^{\frac{-1}{\gamma^2}})^{\gamma} = 1$$

So

$$(e^{\frac{-1}{\gamma^2}})^{-\gamma} = \frac{1}{1}$$

And

$$(e^{\frac{-1}{\gamma^2}})^{-\gamma} = 1$$

The broken function $(e^{\frac{-1}{x^2}})^{-x}$ therefore results in 1 when x tends to 0.

5 Application to the function $(e^{\frac{-1}{x^2}})^{ax}$

We consider:

$$\lim_{x \to \gamma} (e^{\frac{-1}{x^2}})^{ax}$$

To determine a solution for x tending to γ, and therefore 0 (neutral), we will start with:

$$(e^{\frac{-1}{\gamma^2}})^{a\gamma} = ((e^{\frac{-1}{\gamma^2}})^{\gamma})^{a}$$

However

$$(e^{\frac{-1}{\gamma^2}})^{\gamma} = 1$$

So

$$(e^{\frac{-1}{\gamma^2}})^{a\gamma} = 1^{a}$$

So, for all $a \neq \frac{1}{\gamma}$, be included in the interval of all the reals $[\frac{1}{\beta}; \frac{1}{\alpha}]$, ie $[-\infty; +\infty]$,

$$(e^{\frac{-1}{\gamma^2}})^{a\gamma} = 1$$

The broken function $(e^{\frac{-1}{x^2}})^{ax}$ therefore results in 1 when x tends to 0.

Note: The function $(e^{\frac{-1}{x^2}})^{ax}$ having to respect equalities $(e^{\frac{-1}{x^2}}) = ax = 0$ and being $\frac{\gamma}{\gamma}$ equal to 1, a must imperatively be different from $\frac{1}{\gamma}$, otherwise it is not a form 0^0.

6 Generalization

We will generalize our computing methods for all function $u(x)^{v(x)}$, verifying $x = u(x) = v(x) = 0$.
For that, we will study the possibilities of construction and behavior of the functions $u(x)$ and $v(x)$.
First, let's study the function $v(x)$ with respect to x.
We will consider $v(x)$ as:

$$\begin{cases} v(x) = (ax+b)^k \\ v(x) = x \end{cases}$$

If

$$x = \gamma$$

So

$$v(x) = \gamma$$

And

$$\begin{aligned} (a\gamma+b)^k &= \gamma \\ (a\gamma+b)^k(a\gamma+b)^{-k} &= \gamma(a\gamma+b)^{-k} \\ (a\gamma+b)^k(a\gamma+b)^{-k+1} &= \gamma(a\gamma+b)^{-k+1} \\ (a\gamma+b) &= \gamma(a\gamma+b)^-1k-1 \\ (a\gamma+b) &= \frac{\gamma}{(a\gamma+b)^{k-1}} \end{aligned}$$

By reasoning with the absurd, we will dismiss the hypothesis $k = 0$. Indeed, if $k = 0$, then

$$(a\gamma+b) = \frac{\gamma}{(a\gamma+b)^1}$$

And

$$\frac{(a\gamma+b)}{(a\gamma+b)} = \gamma$$

But $\gamma = 0$, implying :

$$\frac{b}{b} = 0$$

But, if b is included $]-\infty; 0[$ or $]0; +\infty[$, we have

$$\frac{b}{b} = 1$$

If $b = 0$, we have

$$\frac{b}{b} = \frac{\gamma}{\gamma} = 1$$

If $b = \frac{1}{\beta}$, ie $+\infty$, we have

$$\frac{b}{b} = \frac{\beta}{\beta} = \frac{-\alpha}{\beta}$$

But, as seen in introduction, we know that

$$\frac{\alpha}{\beta} = -1$$

So

$$\frac{b}{b} = -(-1) = 1$$

In the same way, if $b = \frac{1}{\alpha}$, ie $+\infty$:

$$\frac{b}{b} = \frac{\alpha}{\alpha} = \frac{\alpha}{-\beta} = -(-1) = 1$$

So, for $b \in [-\infty; +\infty]$:

$$\frac{b}{b} = 1$$

We can confirm inequality :

$$k \neq 0$$

Which allows to continue :

$$(a\gamma + b)^k = \gamma$$
$$(a\gamma + b) = \gamma^{\frac{1}{k}}$$

But

$$\gamma^{\frac{1}{k}} = O^{\frac{1}{k}}$$

So

$$\gamma^{\frac{1}{k}} = O^{\frac{1}{k}} = 0 = \gamma$$

Implying

$$a\gamma + b = \gamma$$

And

$$\begin{cases} b = 0 \\ a \in \mathbb{R} \end{cases}$$

Allowing to determine that for all functions $u(x)$ et $v(x)$ verifying $x = u(x) = v(x) = 0$, $u(x)^{v(x)}$ will always be simplyfied, for $x = \gamma$, by the form:

$$u(\gamma)^{v(\gamma)} = u(\gamma)^{\gamma}$$

Implying

$$\begin{aligned}
u(\gamma)^{v(\gamma)} &= u(\gamma)^{\alpha+\beta} \\
u(\gamma)^{v(\gamma)} &= u(\gamma)^{\alpha} u(\gamma)^{\beta} \\
u(\gamma)^{v(\gamma)} &= u(\gamma)^{\alpha} u(\gamma)^{-\alpha} \\
u(\gamma)^{v(\gamma)} &= \frac{u(\gamma)^{\alpha}}{u(\gamma)^{\alpha}}
\end{aligned}$$

But, by generalizing $u(x)$ by that way:

$$u(x) = (qx + p)^r$$

For

$$\begin{cases} q \in \mathbb{R} \\ p \in \mathbb{R} \\ r \in \mathbb{R} \end{cases}$$

We will get:

$$\begin{aligned}
u(\gamma)^{v(\gamma)} &= \frac{((q\gamma + p)^r)^{\alpha}}{((q\gamma + p)^r)^{\alpha}} \\
u(\gamma)^{v(\gamma)} &= (\frac{(q\gamma + p)^r}{(q\gamma + p)^r})^{\alpha} \\
u(\gamma)^{v(\gamma)} &= ((\frac{(q\gamma + p)^1}{(q\gamma + p)^1})^r)^{\alpha}
\end{aligned}$$

So, with $\gamma = 0$, we deduct:

$$u(\gamma)^{v(\gamma)} = ((\frac{p^1}{p^1})^r)^{\alpha}$$

If p is included $]-\infty; 0[$ or $]0; +\infty[$, we have

$$\frac{p}{p} = 1$$

If $p = 0$, we have

$$\frac{p}{p} = \frac{\gamma}{\gamma} = 1$$

If $p = \frac{1}{\beta}$, ie $-\infty$, we have

$$\frac{p}{p} = \frac{\beta}{\beta} = \frac{-\alpha}{\beta}$$

But, as seen in introduction, we know that

$$\frac{\alpha}{\beta} = -1$$

So

$$\frac{p}{p} = -(-1) = 1$$

In the same way, if $p = \frac{1}{\alpha}$, ie $+\infty$:

$$\frac{p}{p} = \frac{\alpha}{\alpha} = \frac{\alpha}{-\beta} = -(-1) = 1$$

Also, for $p \in [-\infty; +\infty]$, ie for all p, we can conclude that

$$\begin{aligned}
u(\gamma)^{v(\gamma)} &= ((\frac{p^1}{p^1})^r)^\alpha \\
u(\gamma)^{v(\gamma)} &= (1^r)^\alpha \\
u(\gamma)^{v(\gamma)} &= 1^\alpha \\
u(\gamma)^{v(\gamma)} &= 1
\end{aligned}$$

So all function $u(\gamma)^{v(\gamma)}$, verifying $x = u(x) = v(x) = 0$, results in 1.

7 Conclusion

Study of broken functions $u(\gamma)^{v(\gamma)}$, verifying $x = u(x) = v(x) = 0$ (neutral zero), by combination of pseudo-limits, respecting

$$\begin{cases}
\alpha = \lim\limits_{h \to 0^+} 0 + h \\
\beta = \lim\limits_{h \to 0^-} 0 + h \\
\gamma = \alpha + \beta = 0
\end{cases}$$

Allows to conclude that, so far indeterminate form, 0^0 is equal to 1.

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